Ten compactness properties of circles: measuring shape in geography

SHLOMO ANGEL
Robert F. Wagner Graduate School of Public Service, New York University, 295 Lafayette Street, New York, 10012-9604 USA (e-mail: solly.angel@gmail.com)

JASON PARENT
Center for Land Use Education and Research, Department of Natural Resources and the Environment, University of Connecticut, Storrs, Connecticut, 06269 USA (e-mail: jason.parent@uconn.edu)

DANIEL L. CIVCO
Center for Land Use Education and Research, Department of Natural Resources and the Environment, University of Connecticut, Storrs, Connecticut, 06269 USA (e-mail: daniel.civco@uconn.edu)

This essay sheds new light on the meaning and measurement of compactness—one of the most intriguing and least-understood properties of geographic shapes. We articulate a unified theoretical foundation for the study of shape compactness that rests on two simple observations: First, that the circle is the most compact of shapes. And second, that there are 10—and possibly more—distinct geometrical properties of the circle that make it the most compact of shapes. We introduce these 10 properties, illustrate them with real-world examples and define indices associated with these properties that can be calculated using a geographic information system.

Key words: circle, compactness, landscape metrics, morphology

Dix propriétés de compacité du cercle : La mesure de forme en géographie

Cet essai apporte une perspective nouvelle à la signification et à la mesure de la compacité spatiale – une des propriétés parmi les plus fascinantes et les moins comprises des formes géographiques. Nous développons les fondations d'une théorie unifiée pour l'étude de la compacité géographique des formes qui reposent sur deux observations simples : premièrement, que le cercle est la plus compacte des formes, et deuxièmement, qu'il y a au moins dix propriétés géométriques distinctes du cercle qui font de celui-ci la forme la plus compacte. Nous décrivons ces dix propriétés, nous les illustrons par des exemples du monde réel et nous définissons des index associés à ces propriétés qui peuvent être calculés en utilisant un logiciel de SIG (Système d'information géographique).

Mots clés : cercle, compacité, indice de forme, morphologie

The Canadian Geographer / Le Géographe canadien 54, no 4 (2010) 441–461
DOI: 10.1111/j.1541-0064.2009.00304.x

Introduction: The Theoretical Framework

Shape matters. The shape of urban footprints matters. The shape of forest patches matters. The shape of election districts matters. The shape of spreading epidemics matters. Some shapes may be more or less efficient, more or less equitable and more or less sustainable than other shapes with the same area. And to understand why that may be so, we need to measure their spatial properties. This article focuses on the measurement of compactness, arguably the most important spatial property of geographic shapes.

Rapid advances in high-resolution satellite imagery and its interpretation have now made it possible to measure geographic shapes in a more rigorous manner. We are presently engaged in a comparative study of a global sample of 120 cities and metropolitan areas based on satellite imagery and census data for two time periods circa 1990 and circa 2000 (Angel *et al.* 2005). Several questions we confronted in this study required metrics for measuring the compactness properties of the urban footprints of these cities: Are cities becoming more circular or more tentacle-like over time, and why? Are some cities more perforated with open space than other cities, and why? Do some cities disturb more open space than other cities, and why? Does the share of vacant urban land remain constant over time? Do cities that are more polycentric discourage transit use? Are trip lengths shorter in more compact cities? Is housing less affordable in more compact cities?

To search for these answers, we have had to embark on a systematic search of the geographic literature for appropriate quantitative measures of the compactness of urban footprints, with the emphasis on the word 'appropriate'. Our search has compelled us to construct a unified theoretical foundation for the study and measurement of shape compactness, to fill an important gap in analytical geography. To the best of our knowledge, no work on shape compactness suggests a methodology that associates discrete measures of compactness with the motivation for measuring them and with the particular problems, processes or concerns they seek to address.

There are indeed a substantial number of studies dedicated to measuring compactness. In the words of one geographer, 'compactness can probably be considered the single most important aspect of geographic shapes. The proliferation of compactness measures in the geographic literature certainly supports this contention' (MacEachren 1985, 65). The study of shape compactness in geography is almost two centuries old. A measure of compactness that compares the perimeter of a given shape to its area was suggested by Ritter in 1822; measures that compare the area of a shape to the area of the smallest circle circumscribing it—as well as to the area of the largest circle inscribed in it—were proposed by Ehrenberg as early as 1892 (Frolov 1975). Over the years, a large number of measures have been proposed and there are several excellent reviews of this literature that introduce them, discuss their properties and evaluate them as well (Blair and Biss 1967; Frolov 1975; MacEachren 1985; Horn *et al.* 1993; Young 1988; Gustafson 1998).

Several authors sense the difficulty in making appropriate selections among the different metrics that have been proposed. They agree that one serious problem that has emerged from the proliferation of measures of compactness is 'confusion concerning the appropriate use of the various measures that have been developed' (MacEachren 1985, 66). They are also concerned that the 'failure to distinguish among aspects of shape opens one to the risks of falling into mathematical malapropisms, in which measures designed to gauge one aspect are applied to another aspect. The results are predictably bad'. (Wildgren 1995, 107)

Dictionaries typically define a compact object as one closely and firmly packed, or as having component parts closely crowded together. That said, how do we determine which one of two abstract shapes on the Euclidean plane is more compact? Do we look for relative roundness as against tentacle-like elongation? Do we look for relative fullness as against perforation or porosity? Do we look for a relatively smooth as against a relatively serrated perimeter? Do we look for a shorter perimeter for a given area? Do we look for a substantial undisturbed core? Do we look for parts of the shape that have strayed the furthest away from each other?

The answers to these questions depend, to a very important extent, on exactly what objects are being represented by these geometric shapes, what forces are acting to form them, what function the shapes appear to perform, and why their compactness should be of concern. Some observers may be concerned with roundness versus elongation, others with fullness versus perforation, and still others with a smooth versus a serrated perimeter or with a shorter perimeter for a given area. By extension, a compactness measure designed to capture roundness may not be able to capture adequately either fullness or smoothness. One designed to capture strategic depth may not be able to capture accessibility. One designed to measure perimeter versus the area circumscribed by city walls may not be able to capture the gerrymandering of election districts.[1]

The aim of this article is to introduce a unified theoretical foundation for the study and measurement of shape compactness, one which takes a discrete type of compactness to be a unique property of a shape, a property which is the likely result of particular forces acting on it. In this endeavour, we follow D'Arcy Thompson, the author of the classic study *On Growth and Form* (1952). Thompson recognized that two objects may take on the same form—say that of a sphere, for example—when different kinds of forces act to shape distinct properties of their respective geometries: 'But the spherical surface of a rain-drop and the spherical surface of the ocean (though both happen to be alike in mathematical form) are two totally different phenomena, the one due to surface-energy and the other to that form of mass energy which we ascribe to gravity' (Thompson 1952, 57)

In fact, as Thompson suggests, the sphere has two distinct geometrical properties. Of all solids of a given volume, it is the shape with the minimum surface area. That is why the raindrop attains a spherical surface area when its surface energy is minimized. The sphere is also the shape with the minimum average distance-squared between all the particles that make it up. That may explain why when all the particles in a star or planet pull each other together they form a sphere. The two properties of the sphere are distinct in the sense that any two shapes that are *not* spheres but have the same volume and the same surface area will not necessarily have the same average distance-squared between the particles that make them up.

The proposed theoretical foundation for the study of shape compactness thus rests on two simple observations: First, that *the circle is the most compact of shapes*. And second, that *there are 10—and possibly more—distinct geometrical properties of the circle that make it the most compact of shapes*.[2] Our emphasis on the distinct geometrical properties of the circle that make it compact was inspired by Hilbert and Cohn-Vóssen's famous essay, 'Eleven Properties of the Sphere' (Hilbert and Cohn-Vóssen 1952). Our work may be seen as a direct application of their groundbreaking insight. When we say that the circle is the most compact of shapes, or, more generally, that a shape is more compact the more it resembles a circle, we seek to specify exactly in what way the shape resembles a circle, or what particular property of the circle it appears to approximate.

The 10 compactness properties of the circle discussed in this essay are, in fact, unique and independent geometrical properties. In the pages that follow, we choose and introduce eleven indices for measuring these 10 properties.[3] In general, there is one index for every compactness property of the circle, selected among several options in the literature, modified, or newly constructed. The 11 indices correspond to 10 propositions concerning the compactness of circles.

Surely, several of these properties are typically found to be highly correlated with each other. Although such correlations are often high, they should be interpreted with care. As we stress throughout this essay, each proposed compactness index measures a 'different' compactness property of the shape and important information

[1] See, for example, the contention of Polsby and Popper (1991, 349) that the perimeter–area relationship 'measures a gerrymander's self indulgence as surely as a breathalyzer measures a drunkard's'.

[2] The circle is also the most *symmetrical* of shapes, but the symmetry properties of shapes are distinct from their compactness properties. They are therefore not essential to this essay.

[3] The additional Exchange Index does not correspond to a unique proposition about the compactness of circles, but does provide a unique way for measuring compactness and is therefore included here.

about this property may be lost when we select an index focusing on another property to represent it.

The 10 compactness properties of shapes that are the focus of this essay are independent. The fact that the circle has zero variability in the distances from its centre to its perimeter, for example, does not imply that the circle, of all possible shapes, has the shortest perimeter for a given area. The fact that a circle has the shortest perimeter for a given area does not imply that of all possible shapes, the points in a circle are closest to its centre. Nor does it imply that they are also closest to each other. Each one of these properties is a unique geometrical property of the circle that requires a separate mathematical proof to assert it.

The search for a single metric or a composite metric that fully characterizes shape compactness in a manner that always corresponds to our intuitions of what makes shapes compact is doomed to fail. In fact, it has failed,[4] as well it should have. In our view, it is much more important to measure a single compactness property that captures the essence of what really needs to be measured than to 'hedge our bets', so to speak, by measuring everything and taking an average.

Given the wide choice of possible compactness properties than can now be measured, we can—and indeed must—focus on the particular property that needs to be measured before proceeding to measure anything. We should ask ourselves what is the underlying function that the shape appears to perform or needs to perform, or what are the underlying forces acting on that shape to give it form. With that in mind, we can confidently proceed to select the property that best corresponds to the function that the shape seeks to perform. The discovery of relations between form and function can be traced back to Aristotle, but is best captured in a quote by Louis Sullivan, one of the pioneers of modern architecture: 'It is the pervading law of all things organic and inorganic, of all things physical and metaphysical, of all things human and all things super-human, of all true manifestations of the head, of the heart, of the soul, that the life is recognizable in its expression, that form ever follows function. *This is the law*'. (Sullivan 1896, 408).

Each compactness property of a given shape needs to be understood in terms of its underlying function or purpose. If no function or purpose can be discerned or stipulated, then it should not matter at all whether the shape is compact or not. There is no sense in measuring shape attributes that are irrelevant to a specific investigation of compactness that has some purpose in mind. In the pages that follow, we discuss each one of these 10 compactness properties and their associated indices by focusing on the role of function and force in the formation of shapes in the real world.

In constructing the indices we adhere to five rules:

1. The index must correspond to *a recognizable property* of the shape that is associated with a recognizable function or set of forces.
2. There must be *real-world examples* that illustrate this property—as well as its associated function or set of forces—at both the low end and the high end of the index.
3. The index must apply to *all two-dimensional geometric shapes*, including those made up of several non-contiguous patches.
4. The index must be *dimensionless* (independent of the size of the shape) as well as *directionless* (independent of its orientation).
5. The index must *vary between 0 and 1*, with the value of 1 assigned to the circle as the shape with maximum compactness.

The 10 distinct compactness properties of shapes presented here, taken together, provide a comprehensive—but, as yet, possibly incomplete—perspective on what constitutes shape compactness in the broadest sense. In the following eleven sections we focus on each of these properties individually. In each section, we define an index and give several real-world examples of shapes that seek to attain a high score or a low score on this index. For the reader interested in employing these indices in practice, a mathematical appendix shows how each compactness index can be derived and calculated in mathematical and geographical information system (GIS) terms.

[4] Horn *et al.* (1993, 113), for example, reviewed several comparative studies of compactness and concluded that no single measure always gives values that best agree with our intuitive perceptions of shape compactness.

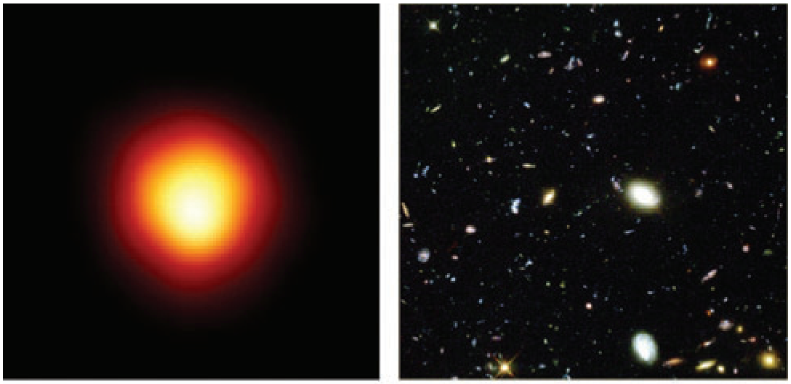
(A) (B)

Figure 1
(A) Cohesion compactness in the Betelgeuse star (high). (B) Cohesion compactness in the distribution of matter in our expanding universe. (low)
SOURCE: (A) Dupree, A. (CfA) and Gilliland, R. (STScI) 1999 'Hubble telescope image', 5 June, NASA. (B) Williams, R. and the HDF Team (STScI) 2000 'Galaxies Observed in the Hubble Deep Field', 9 July, NASA.

Cohesion

Cohesion is a natural measure of the overall accessibility of decentralized, polycentric metropolitan areas, for example. The cohesion property of geographic shapes focuses attention on the closeness of everything to everything else. The more cohesive these metropolitan areas are, the better access to all the opportunities within them they offer their inhabitants. More generally, cohesion is a natural measure of the compactness of all geographic shapes that are non-hierarchical: a set of small islands, a set of homesteads or a set of transit stations.

- The ***Cohesion Proposition***: Among all shapes of a given area, the circle has the shortest distance-squared[5] among all its points.
- The ***Cohesion Index*** is the ratio of the average distance-squared among all points in an equal-area circle and the average distance-squared among all points in the shape.

To the best of our knowledge, there is no allusion to this metric in the geographic compactness literature.

[5] It stands to reason that among all shapes of a given area the circle also has the shortest average *distance*, rather than distance-squared, among all its points, but a rigorous proof of this proposition has so far eluded us.

The fact that the two-dimensional circle has maximum cohesion compactness can be readily observed by looking at a three-dimensional sphere. Stars have a spherical shape because the particles of plasma of which they are made gravitate towards each other, seeking to get as close as possible to each other. In a spherical star (Figure 1A), the average distance-squared between all particles is at a minimum.

People typically congregate in villages and towns because they want to be close to each other and to maximize contacts and exchanges among them. They 'gravitate' towards each other to create cohesive communities. When everyone wants to live and work as close as possible to everyone else, such settlements often attain compact forms that resemble circles.

In contrast, a few couples on a secluded beach may seek to be as far as possible from each other to maximize their privacy. Similarly, the cohesion compactness of matter in our expanding universe is reduced by galaxies moving away from each other at speeds close to the speed of light (Figure 1B).

Proximity

The proximity property of geographic shapes focuses on the distance of the entire shape to its

Figure 2
(A) Proximity Compactness in the circus tent (high). (B) Roman Empire at its peak territorial extent—second century C.E. (low)
SOURCE: (A and B) Hanson, Dwight, n.d. 'Cannobio Big Tent'. Photo courtesy of Hanneford Circus Inc.

centre, however defined. In monocentric cities, for example, proximity is a natural measure of accessibility of a metropolitan area to its Central Business District (CBD). More generally, proximity is a natural measure of the compactness of all geographic shapes that emanate from a centre.

- The ***Proximity Proposition***: Among all shapes of a given area, the circle has the shortest average distance to its centre.

Because there are many possible centres for a given shape,[6] we must define at least one relevant centre of the shape before defining the Proximity Index:

- The ***Proximate Centre*** is the centre of gravity of a shape[7]
- The ***Proximity Index*** is the ratio of the average distance from all points in the equal-area circle to its centre and the average distance to the Proximate Centre from all points in the shape.

Thünen proposed an index 'represented by the mean distance between elements of a figure to an arbitrary point (the so-called mean distance from field to farmhouse)' in 1842 (Thünen 1966 in Frolov 1975, 684). Frolov observed that the mean distance to the centre of a circle is equal to two-thirds of its radius and suggested that it would be 'quite logical' to use it for the construction of measures of compactness (Frolov 1975, 685). The Proximity Index employs Frolov's suggestion.

Circus tents have a circular perimeter precisely to embody this compactness property of circles, seeking to have everybody as close to the action as possible (Figure 2A).

Some phenomena tend to reduce, rather than increase, their proximity compactness. Agricultural and cultural innovations, as well as epidemics, may spread from a given point in different directions at different rates. The Proximity Index may then measure the degree to which a shape has expanded away from its origin in an uneven way. Ancient Rome, for example, situated at the centre of the Italian Peninsula, was a city-state until the fourth century BC when it conquered the Italian Peninsula and began to expand outwards in a series of conquests that established the Roman Empire, attaining it widest extent some six centuries later (Figure 2B). The Empire ruled over vast territories, some closer and some very far away from the City of Rome—its centre—giving it a relatively low value on the Proximity Index.

[6] See discussion of centres of different types in de Smith *et al.* (2007, 79–84).
[7] This centre is defined as the centroid or center of gravity in de Smith *et al.* (2007, 83).

Figure 3
How exchange creates noncompactness
NOTE: When no credit is given, photos, figures and maps were either drawn by the authors or are in the public domain. All URLs were accessed on 25 October 2009.

Exchange

The exchange property of geographic shapes measures how much of the area inside a circle is exchanged with the area outside it to create a given shape (Figure 3). The essence of political gerrymandering, for example, is the creation of electoral district boundaries that jettison nearby voters—those living near the centre of a given district—in exchange for voters living further away, in order to affect election outcomes for political gain. Exchange is thus a natural metric for detecting aggressive gerrymandering.

Exchange compactness is not a unique compactness property of circles and there is thus no exchange proposition. The Exchange Index, which is a unique way of measuring compactness, makes use of the fact that the equal-area circle is the most compact of shapes in terms of cohesion and proximity.

- The ***Exchange Index*** is the share of the total area of the shape that is inside the equal-area circle about its Proximate Centre.

The use of the Exchange Index, as defined here, as a measure of compactness is described in Whittington *et al.* (1972, 12–13). The Exchange Index is a rather crude measure of compactness. It does indeed look at all points in a given shape, but it only seeks to determine whether they are within its associated circle or not. The index does not distinguish how close or how far they are to the circle's centre.

Justices on the US Supreme Court, when discussing gerrymandering, often allude to its voter exchange aspect: 'A district that *reaches out* to grab small and apparently isolated minority communities is not reasonably compact' (US Supreme Court 2006a); 'specific *protuberances* on the draconian shape that *reach out* to include Democrats, or *fissures* in it that squirm away from Republicans' (US Supreme Court 2004); 'incorporated multiple, small, *farflung pockets* of minority population' (US Supreme Court 2006b); 'its "many narrow corridors, wings, or fingers… *reach out* to enclose black voters, while excluding *nearby Hispanic residents*"' (US Supreme Court 1996) (italics ours). Blatant and unrestrained voter exchange typically results in bizarre district shapes that have very low Exchange Index values.

Figure 4 shows the shapes of three US Congressional districts in 2004. District 25 in Texas, for example, was found to have one of the lowest scores[8] on the Exchange Index—0.3 (Figure 4A). District 13 in Ohio (Figure 4B) had exactly 50 percent of its shape within its corresponding equal-area circle, and therefore a score of 0.5 on the Exchange Index. District 4 in Arizona (Figure 4C) had the highest score on the index—0.9.

Using the Exchange Index in conjunction with several others, our initial calculations suggest, for example, that the gerrymandering of US Congressional districts is now more severe than before, and that election districts are considerably more gerrymandered in the United States than in the United Kingdom.

[8] Most of the districts with even lower scores on the Exchange Index are split apart by bodies of water. The Exchange index, in the form it is presented here, does not account for separation by bodies of water.

Figure 4
Exchange Compactness in US Congressional Districts in 2004: (A) Texas Dist. 25 (0.3, low); (B) Ohio Dist. 13 (0.5); and (C) Arizona Dist. 4 (0.9, high)

Perimeter

The perimeter property of geographic shapes focuses on the compactness of their outer boundaries. The perimeter index is a natural measure of the compactness of a walled city, for example. It is also a natural measure of the sustainability of small forest patches: the shorter their perimeter relative to their area, the less disturbed they are by adjoining land uses.

- The ***Perimeter Proposition***: Among all shapes of a given area, the circle has the minimal length of contact with its periphery.
- The ***Perimeter Index*** is the ratio of the perimeter of the equal-area circle and the perimeter of the shape.

The ratio of perimeter to area as an important attribute of geographic shape was recognized by Ritter as early as 1822 (Ritter 1822 in Frolov 1975, 678). The particular form of the Perimeter Index used here was formulated by Nagel in 1835 (Nagel 1835 in Frolov 1975, 679). Numerous other perimeter-area indices have been proposed in the literature over the years (Horn *et al.* 1993; McGarigal and Marks 1995). Several do not conform to the five rules articulated earlier.[9]

In some shapes there may be an interest in maximizing the perimeter for a given cross-sectional area, so as to increase the interaction with its immediate environment as much as possible. The wall of the small intestine (Figure 5A) is crenellated so as to maximize the contact area for absorbing the nutrients in the food passing through it. Hence its cross-section will have a low Perimeter Index value. In some other shapes—a plan cross-section of a gas storage tank is one example (Figure 5B)—there may be an interest in reducing the perimeter of a given area to a minimum. The objective in the case of the gas tank may be to economize on the amount of material needed to construct the vertical walls of the tank.

Conversely, there may be an interest in enclosing as large an area as possible with a perimeter of a given length. This problem has been known since ancient times as Dido's problem. In the *Aeneid*, Virgil recounts the story of Dido, a Phoenician princess who escaped persecution by her brother, arriving in the bay of Tunis on the Mediterranean shore. There she negotiated for a piece of land, only as much land as could be encircled by a bull's hide. She then 'cut the hide into narrow strips, tied them together, and enclosed a large tract of land. On this land she built a fortress and, near it, the city of Carthage' (Tikhomirov 1990, 9). Dido used the seashore as a boundary too, and so the area she enclosed was a semi-circle. That the solution to Dido's problem in the Euclidean plane was a full circle was already known to the ancient Greeks, and the first proof that the circle is the largest area enclosed by a given perimeter is attributed to

[9] The Fractal Dimension, for example—often defined as $F = 2logP/logA$, where P is the perimeter and A is the area of the given shape—is not dimensionless and does not assign values between 0 and 1, with a value of 1 to the circle (Milne 1988).

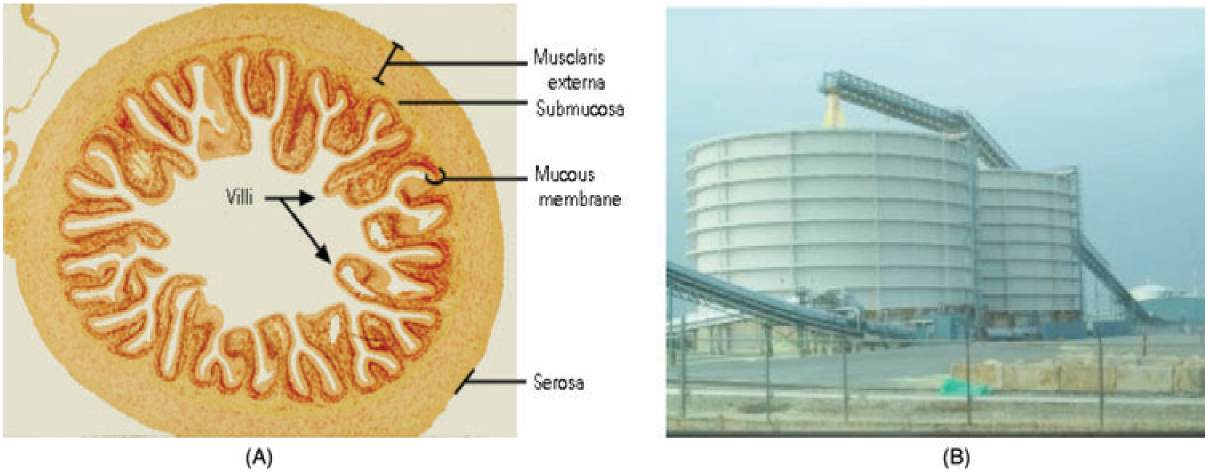

Figure 5
Perimeter Compactness inside the small intestine (low). (B) Perimeter Compactness in a sulphur storage tank wall in Alberta (high) SOURCE: (A) Ackerley, Sandra K., n.d. 'Light micrograph of histological section of mature small intestine' (Guelph, ON: College of Biological Science, University of Guelph). (B) Bulbrook, Daniel J. 2008 'Shell Shantz Plant', 6 May (Available at: http://ontheroadtravels.blogspot.com).

Zenodorus who lived in the second century BC (Tikhomirov 1990, 11–15).

Fullness

The smoother its boundary and the less perforated or porous its interior, the more compact the shape would be in the sense of fullness. Fullness is therefore a natural metric of the relationship between the built-up patches of cities and the open spaces in and around them, for example. It is also a natural measure of forest fragmentation.

- The ***Fullness Proposition***: Among all shapes of a given area, small neighbourhoods around all points in a circle contain the least amount of its periphery.

We define a small neighbourhood, fullness and the fullness index as follows:

- A ***small neighbourhood*** around a given point in a shape is a circle with an area equal to 1 percent[10] of the total area of the shape.
- ***Fullness*** is the share of the area of a small neighbourhood that belongs to the shape, rather than to its periphery.
- The ***Fullness Index*** is the ratio of the average fullness of small neighbourhoods in the shape and in its equal-area circle.[11]

Burchfield *et al.* (2006, 602) find that in the United States, 'on average, 42 percent of the land in a square kilometre surrounding residential development was open space circa 1976'. They define a sprawl index as a measure of absence of built-up area rather than as fullness, and they take a small area of a fixed size (one-square kilometre) as a small neighbourhood surrounding a given point in the shape. The Fullness Index is a variation on their Sprawl Index, taking an area amounting to one percent of the area of the shape as a small neighbourhood. The index is thus independent of the size of the shape being measured.

The Fullness Index, as its name implies, focuses on compactness as the degree to which a given shape does not have any fragments of the periphery in it—such as the holes in a cross section of a piece of Swiss cheese, for example—or

[10] The selection of 1 percent of the total area of the shape as a small neighbourhood is admittedly arbitrary.

[11] An alternative way to measure the fullness of shapes is to examine the frequency that any given pixel in the shape will have a peripheral pixel adjacent to it. We have indeed constructed such a measure. As expected, preliminary calculations on a sample of cities show that it is highly correlated with the Fullness Index. Both measures depend on the relative sizes of small neighbourhoods or pixels.

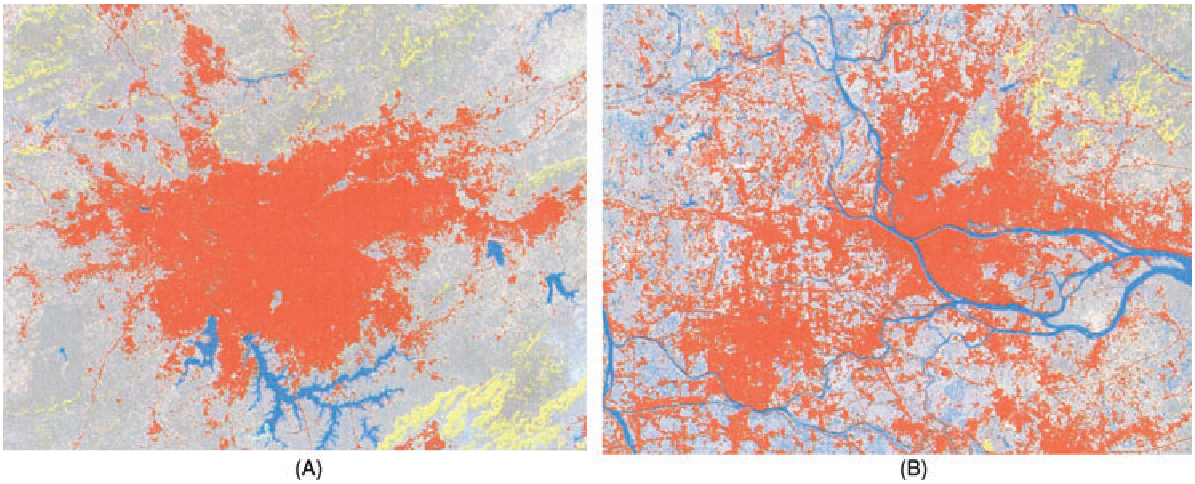

Figure 6
(A) Fullness in the built-up areas of São Paulo, Brazil (high). (B): Fullness in the built-up areas of Guangzhou, China (low)

to the degree that its outer boundary is smooth rather than fragmented and disjointed by the penetration of the periphery into it.

In some cases, say in a circular rubber diaphragm that has to block the passage of any liquid through it, the integrity of the diaphragm requires that it not have any puncture in it at all, even of the smallest size. The Fullness Index of such a diaphragm will therefore be at a maximum. In other cases, say in a chain-link fence that has to block the passage of small animals but let in as much wind as possible to flow through it, there may be an interest in decreasing its fullness to a minimum while preserving an adequate level of strength and rigidity in the fence.

In still other cases, say in urban areas, there may be an interest in maintaining an optimal level of compactness in the sense of fullness. If a city is fully built and its entire area is fully covered with impervious surfaces, its inhabitants will not have adequate access to green open space. This appears to have been the case in São Paulo, Brazil in 2000 (Figure 6A). The Fullness Index of the built-up area of São Paulo in that year was 0.72. In contrast, if the periphery of the city contains villages that accommodate urban workers and businesses while maintaining lands in agricultural use, then the city may be less compact than the desired optimum. Built-up areas will not be contiguous, requiring lengthy and expensive extensions of urban infrastructure and longer average commutes, while fragmented agricultural fields will become less productive. This appears to have been the case in Guangzhou, China in 2000 (Figure 6B). The Fullness Index of the built-up area of Guangzhou in that year was 0.41.

Depth

Depth is a natural measure of the relative threat a country may face because of the shape of its borders or of the relative protection that a forest may offer to those of its inhabitants that need to be away from its edge. The depth property of geographic shapes focuses on the distance of all parts of the shape from its periphery.

- The ***Depth Proposition***: Among all shapes of a given area, the circle has the longest average distance from its interior points to its periphery.
- The ***Depth Index*** is the ratio of the average distance to the periphery in the shape and the average distance to the periphery in its equal-area circle.

Rohrbach proposed as early as 1890 that the mean distance to the periphery be used as a measure of geographic shape (Rohrbach 1890 in

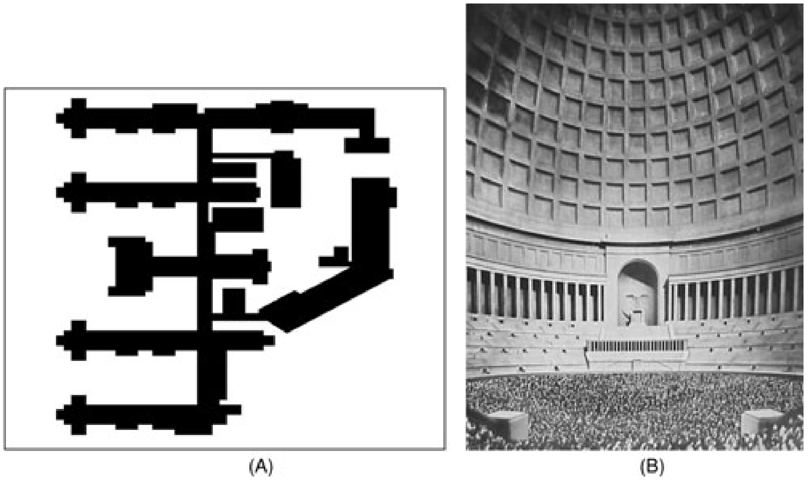

Figure 7
(A) Depth Compactness in a building complex with well-lit rooms (low). (B) Depth Compactness in Speer's 1936 design for the *Volkshalle* (high)

Frolov 1975, 683). Frolov proposed standardizing it, but did not suggest how (Frolov 1975, 684). Standardizing it with the mean distance to the periphery in the equal-area circle makes it into a valuable measure of compactness.

The Depth Index focuses on the observation that shapes that are made up of many long protrusions into their periphery or have many long intrusions into them are not compact. This attribute can best be seen in the relationship between buildings and the open space around them, when they do not rely on air conditioning and artificial light. Residential buildings in particular are made up of a lot of small rooms and it is preferable that most rooms have windows to let in fresh air and natural light. That is why the floor plans of large residential buildings—college dormitories, for example—are almost never circular in shape, allowing the structures and the open space around them to interpenetrate each other (Figure 7A). The land coverage of such a building typically has a low value on the Depth Index because, on average, all its areas are relatively close to its outer periphery.

In contrast, imagine a large, domed, circular hall with doors along its entire perimeter. If a fire broke out and the people had to quickly evacuate the hall by running out through the nearest exit, they would have to run a longer distance, on average, than they would in any other shape hall with the same area. Such a circular hall, like the *Volkshalle*, the monumental domed hall designed by Albert Speer, Hitler's Architect, for Berlin in 1936 (Figure 7B), will therefore have a near maximum value on the Depth Index. The evacuation of the *volk* in case of fire was clearly not a major concern, was it?

Dispersion

Dispersion is a natural way to examine the advancing frontiers of spreading phenomena—be they epidemics, earthquakes or inventions—away from their epicentres. The Dispersion property of a geographic shape focuses on the extent to which the distance from its centre to its perimeter varies in different directions.

- The ***Dispersion Proposition***: Among all shapes of a given area, the circle has the minimum variability in the radial distances from its centre to its perimeter.

Indeed, the fact that the points on the perimeter of the circle are all equidistant from its centre is the most familiar property of the circle, and typically constitutes its very definition. The circle thus exhibits no variability at all in the distances from points on its circumference to its

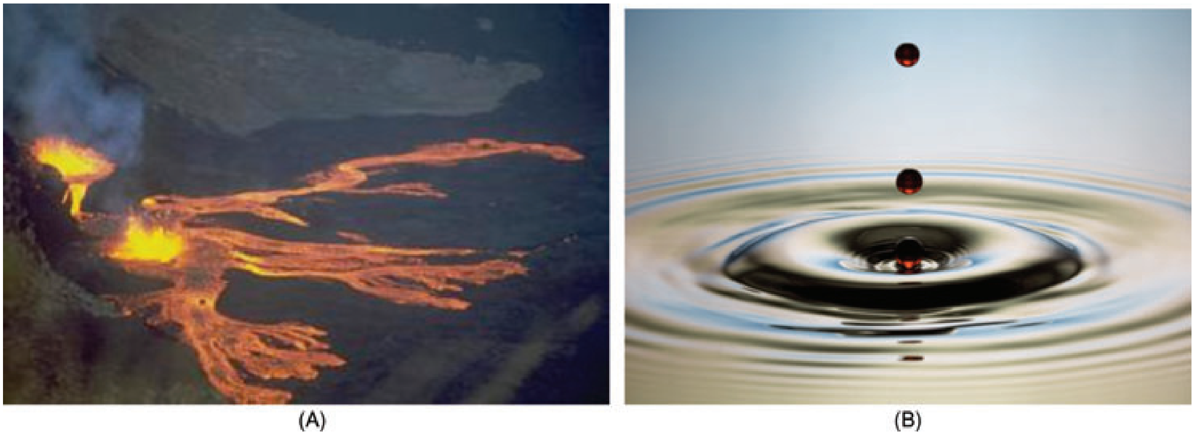

Figure 8
(A) Dispersion Compactness of lava from Fernandina Volcano, Galapagos Islands, 1978 (low). (B) Dispersion Compactness in a ripple created by a falling drop of water (high)
SOURCE: (A) Orbach, Marc 1978, 'Fernandina Volcano, Galapagos Islands' (Available at: http://www.volcano.si.edu/world/volcano.cfm?vnum=1503-01=&volpage=photos&photo=046081). (B) Waugh, Martin, n.d. 'Three Red Balls', (Available at: www.liquidsculpture.com).

centre. We can define an index that compares the compactness of any shape to that of a circle with the same average distance of points on its perimeter to its centre. We define this circle and the Dispersion Index as follows:

- The ***Average-Distance Circle*** is a circle about the centre of a given shape with a radius equal to the average distance of points on the perimeter of the shape to its centre.
- The ***Dispersion Index*** of a given shape is the ratio of the area of the shape inside the Average-Distance circle and the area of the Average-Distance Circle.

This index is a continuous formulation of the Boyce–Clark Index (Boyce and Clark 1964) that has been widely applied over the years. The original formulation involved a limited set of points on the perimeter of the shape—its intersections with a set of equally spaced radials.

Clearly, the closer the perimeter of the shape is to the perimeter of its associated Average-Distance Circle, the more compact the shape in terms of dispersion. The Dispersion Index needs to be distinguished from the Proximity Index defined and discussed earlier. The Proximity Index focuses on the shape *as a whole* and measures distances from *all* points in the shape to its centre. The Dispersion Index focuses only on the perimeter of the shape—its edge, its frontier, or, more generally, its contact with the periphery—and on the distances of points on this perimeter to the centre. In other words, it completely ignores the points that constitute the interior of the shape.

In some shapes what is of critical interest is the frontier, rather than the bulk of the shape itself, and—more specifically—the frontier in relation to an identified centre. Lava coming down from an erupting volcano, for example, moves away from the mouth of the volcano at different speeds in different directions, largely depending on the topography of the mountain and it surrounding ridges and valleys. This can be clearly seen in the image of advancing lava from the eruption of the Fernandina volcano in the Galápagos Islands in 1978 (Figure 8A). How far it has reached in any direction is of critical interest to villagers and townspeople living nearby. Storm fronts, military conquests, floods, epidemics, inventions, rumors and our expanding universe also advance at different speeds in different directions from an original, historical centre—in the latter case, the location of the Big Bang—and their dispersion may be appropriately measured with the Dispersion Index.

There are some phenomena, however, that advance at almost equal speeds in all directions.

Waves moving through a uniform medium—be they water waves, sound waves or earthquake shock waves, to take a few examples—propagate in all directions with equal speed. This is why ripples on a pond, for example, tend to form concentric circles (Figure 8B).

Range

Range is a natural metric for measuring the extent to which straying parts of the shape have ventured away in different directions, say naval vessels in an armada sailing through thick fog. The Range property of a geographic shape focuses on the individual points on its outer perimeter that are the furthest away from each other.

- The ***Range Proposition***: Among all shapes of a given area, the circle has the shortest distance between its furthest points.
- The ***Range Index*** of a given shape is the ratio of the diameter of its equal-area circle and the diameter of the smallest circle fully circumscribing the shape.

Ehrenburg proposed this index—in the form of the ratio of radius of the smallest circle circumscribing the shape and radius of the equal-area circle—as a measure of geographic shape in 1892 (Ehrenburg 1892 in Frolov 1975, 681).

Unlike many of the preceding indices, the Range Index is not an *average* measure. The index focuses attention on the distance between the furthest edges of a given shape, and is therefore subject to the undue influence of small patches that are part of the shape but far away from the rest of the shape. In this sense, it may indeed depend on two extreme points and may not be a very good intuitive measure of the overall compactness of a given shape.

Small tribes in the Papua New Guinea highlands, for example, have been living in relative isolation from each other with little interest in contact, commerce or communication between them. Each tribe has developed its own language, and that language is spoken only in the small area—typically an isolated valley—occupied by the tribe. Many of the language areas in this region are rather small. Most of these language areas have simple shapes that are not at all intertwined with each other, and therefore have a relatively high level of dispersion compactness (Figure 9A).[12] In contrast, the distribution of cases of Avian Flu in Thailand in 2005 was highly dispersed, with some very small areas separated by great distances from the rest. The virus has propagated as far and wide as possible, resulting in a relatively low value of range compactness (Figure 9B).

Girth

The Girth property of geographic shapes is a measure of the thickness of the layer insulating its innermost core from its periphery. Girth is a natural metric for measuring the extent to which large patches of the Amazon rainforest, for example, are still able to insulate indigenous tribes from contact with Western civilization.

- The ***Girth Proposition***: Among all shapes of a given area, the circle has the thickest layer insulating its innermost point from its periphery.
- The ***Innermost Point*** of a shape is the point that is furthest away from its periphery.[13]
- The ***Girth Index*** is the ratio of the thickest layer insulating the innermost point of the shape from its periphery and the radius of the equal-area circle.

Ehrenburg proposed this index too—in the form of the ratio of radius of largest circle inscribed in the shape and the radius of the equal-area circle—as a measure of geographic shape in 1892 (Ehrenburg 1892 in Frolov 1975, 681).

Like the preceding Dispersion Index, the Girth Index is not an 'average' measure. The index focuses attention on the singular point that is furthest from its periphery, and is therefore subject to the undue influence of a small part of the shape that may have a substantial core. In this sense, it may indeed depend on some extreme point that is much more isolated from the periphery than a typical point in the shape. In this sense the Girth Index, like the preceding Range Index, may not be a very good intuitive measure of the overall compactness of a given shape.

[12] Because we use the convention that all indices assign the highest level of compactness to the circle, a low level of actual dispersion will imply a high level of *dispersion compactness* and vice versa.

[13] This point may not be unique to circles, but that does not affect the analysis.

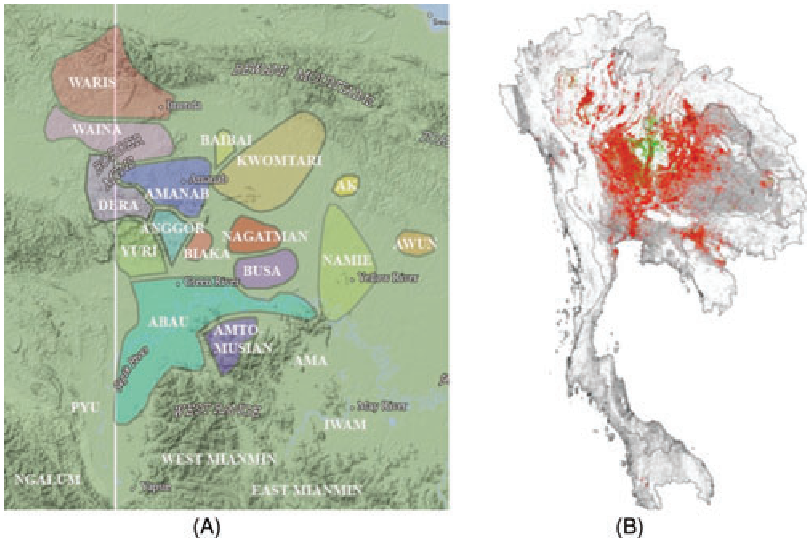

Figure 9
(A) Range Compactness in Papua New Guinea language areas (high). (B) Range Compactness in the distribution of cases of avian flu in Thailand, 2005 (low)
SOURCE: (A) Craig, Barry, 2009 'Language areas of the Upper Sepik (or Central New Guinea) region'. Based on Wurm & Hattori 1981, Map 6, and Google Earth terrain images. (B) Butler, Declan 2005 'Drugs could head off a flu pandemic—but only if we respond fast enough'. Reprinted by permission of *Nature* Magazine, 436, 614–615.

Some shapes, like threads, snakes or earthworms, may seek to minimize girth compactness, so that they can pass through very narrow passages. Other shapes, like roads, high-voltage wires, oil pipelines or canals may seek to minimize girth compactness so as to cover as long a distance as possible with widths or cross-sections that are as narrow as permissible. The Corinth canal in Greece, for example, was built with minimal girth throughout, so as to allow the passage of ships of a certain girth while minimizing excavation (Figure 10A). As a shape, therefore, it has very low girth compactness.

In contrast, some shapes may seek to increase their girth compactness. A knot is often tied at the end of a thread used for embroidery to increase the girth of the thread, preventing its end from passing through the fabric. A flooding river increases its girth to cover as wide an area as possible. The African egg-eating snake seeks to expand the girth of its body to swallow as big an egg as possible. The internal organs of the *Diadema antillarum* sea urchin form its core and give it a high value on the Girth Index (Figure 10B), while its long spines give it a low value on the Range Index. Geometrically, the girth of a given shape as defined here is actually the radius of the largest circle fully contained in the shape.

Traversal

We may ask ourselves whether the cultivation of olives along the entire perimeter of the Mediterranean since biblical times may be due, in part, to the relative ease of traversing it in all directions. The Traversal property of geographic shapes focuses on the ease of connecting all the points along their perimeter to each other.

- The ***Traversal Proposition***: Among all shapes of a given area, the circle has the shortest average distance along interior paths between points on its perimeter.
- The ***Traversal Index*** is the ratio of the average distance between all points on the perimeter of the equal-area circle and the average distance along interior paths between all points on the perimeter of the shape.

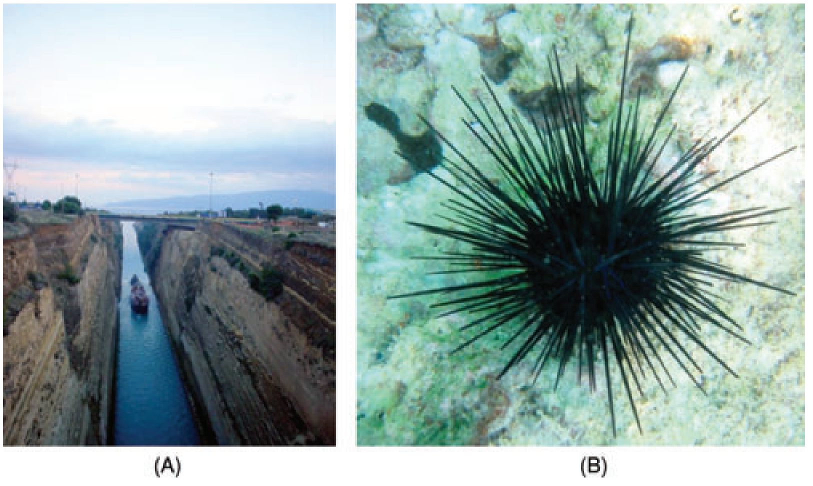

Figure 10
(A) Girth Compactness in the Corinth Canal in Greece (low). (B) Girth Compactness of the *Diadema antillarum sea urchin* (high)
SOURCE: (A) Anagnostopoulos, Robert, n.d. 'Corinth Canal'. Courtesy of Ralo Inc. (Available at: www.myolivetrees.com). (B) Division of Fish and Wildlife, US Virgin Islands 2003 'Diadema Antillarum Sea Urchin' (Available at: http://fw.dpnr.gov.vi/education/FactSheets/PDF_Docs/13Diadema.pdf).

To the best of our knowledge, there is no reference in the available literature to the Traversal Index—or to any similar index that focuses on the average length of interior paths connecting points along the perimeter of a given shape—as a measure of compactness.[14]

People who sit in a meeting in a large room seek to be as close to each other as possible while being able to see each other face-to-face without anyone blocking their view. Shoulder-to-shoulder, a given number of people sitting in a meeting will have a total perimeter of a certain length. Of all the possible sitting arrangements, the circle offers them the best possibility of being closest to each other and having an unobstructed view of each other. This is why people meeting as equals typically arrange themselves into a circle, like the circle of hockey players shown in Figure 11A.

In contrast, some phenomena may seek to reduce traversal compactness. In a maze, for example, the interior paths between points on the perimeter are very long but no shortcuts across its walls are allowed (Figure 11B). Such a shape will have a very high value on the Proximity Index and on the Cohesion Index, but very low values on the Traversal Index and on the Perimeter Index.

[14] Kendall and Moran (1963, 9–10) point out that there are a number of different ways to define the length of a random chord in a circle, each one yielding a different average. We have selected one of them to construct this index.

Detour

The near-circular island of Mauritius is easier to sail around than the long and thin island of Cuba. The detour property of geographic shapes focuses on the relative difficulty of circumnavigating them when they are viewed as impassable obstacles.

- The ***Detour Proposition***: Among all shapes of a given area, the circle offers the minimum ease of circumnavigating it from any direction.

Before defining a detour index, we first define the Convex Hull:

- The ***Convex Hull*** of a given shape is the convex polygon of the shortest possible perimeter that fully encompasses it.

An easy way to visualize the Convex Hull of a shape is to place a rubber band around the shape and release it. The rubber band then

(A) (B)

Figure 11
(A) Traversal Compactness in a circle of hockey players, Team Canada, Women's World Hockey Championships (high). (B): Traversal Compactness in a maze (low)
SOURCE: (A) Canadian Press/Frank Gunn 2008 'Team Canada Players Have A Circle Meeting/Women's World Hockey Championships' (Available at: http://www.ctvolympics.ca/hockey/news/newsid=7189.html). (B) Brindle, John, n.d. 'Shefford Maze' (Available at: http://wuff.me.uk/hoo%20hill%20maze/noj.html).

(A) (B)

Figure 12
(A) Detour Compactness in the Great Wall of China (low). (B) Detour Compactness in Place Charles de Gaulle in Paris (high)
SOURCE: (A) Anthony Willoughby n.d. (Available at: www.changemakerweb.co.uk). (B) Place Charles de Gaulle, Paris, Google Earth.

forms the Convex Hull of the shape. The simplest way to define a Detour Index is as follows:

- The ***Detour Index*** of a given shape is the ratio of the perimeter of its equal-area circle and the perimeter of its Convex Hull.

To the best of our knowledge, there is no allusion to this metric in the geographic compactness literature.

The Detour Index focuses on the shape as an obstacle to free movement, movement in a straight line, for example, from a point on one side of the shape to a point on its other side. It assumes that movement across the shape is not permissible or possible. It is therefore sensitive to the location of points on the outer perimeter of the shape, but it is oblivious to points inside it.

Obstacles that are difficult to circumvent—both natural and artificial—are typically linear in shape. The Great Wall of China, the longest artificial obstacle, is one such example (Figure 12A). Other artificial obstacles include firebreaks separating parts of a forest from each other, fences, hedges and trenches. Natural obstacles that are difficult to circumvent include gorges, mountain ridges and prominent natural formations like the Florida Keys, or the Outer Banks of North Carolina. A circular clearing in a forest is less disturbing to its integrity as a wildlife habitat than a long swath cut through it, because it is easier to avoid. Typically, the longer and narrower the obstacle, the more difficult it is to circumvent, and the lower is its score on the Detour Index.

In contrast, the more a shape resembles a circle, the easier it is to circumvent. Cars traveling on the nine avenues that converge on Place Charles de Gaulle in Paris, for example, easily traverse it by circumventing the circle on which the Arc de Triomphe is situated (Figure 12B). They veer only minimally from the path they would have followed in the absence of this obstacle, and any other shape with the same area would have offered a more forbidding obstacle.

Conclusion

If we are to make any systematic comparison or to test any statistical hypothesis concerning the compactness of geographic shapes we need to reduce it to a metric. Since there are many possible metrics to choose from, the problem is to identify the right one. We can begin to solve this problem by noting that the compactness of the circle, the most compact of shapes on the plane, is not a single property but rather a bundle of properties.

It is useful to imagine that acting upon any one of these properties, while ignoring all others, can transform a shape into a circle of the same area. If we take a shape and increase its *cohesion*—bringing its parts closer together—it will gradually become a circle. If we take a shape and increase its *proximity*—bringing its parts closer to its centre—it will gradually become a circle. If we move parts of the shape around to reduce its overall *perimeter,* it will gradually become a circle. If we increase the *fullness* of the shape, it will gradually become a circle. If we reduce the *range* of a shape—bringing in distant points so that no point is too far away—it will gradually become as circle. If we increase the *girth* of the shape—pushing out the parts of the perimeter that are closest to its innermost point—it will gradually become a circle. If we reduce *detour* in a shape—tightening its convex hull—it will gradually become a circle.

Geographic shapes may often be a focus of policy, and we want to make them more compact or less compact to attain some societal objective. We are concerned with the compactness of cities, for example, both to the extent that it facilitates or hinders accessibility and to the extent that it fragments the open space in and around it. If only accessibility to the CBD matters, then the Proximity Index may be the natural way of measuring compactness. If accessibility of the entire city to itself matters, then the Cohesion Index may be a more appropriate measure. The Fullness Index will be better than either of these indices to measure the fragmentation of open space. The Perimeter Index will be of little or no use in measuring accessibility and of minimal use in measuring fragmentation. It will be of no use at all in measuring the extent of gerrymandering of election districts. The Exchange, Cohesion and Proximity indices are much better at that. And when it comes to the protection of wildlife in forest patches, the Depth Index may provide a better metric than any of these indices. The Perimeter Index will also help in identifying the extent of disturbance of forest patches by nearby land uses.

In conclusion, the better we understand the properties of geographic shapes and the forces acting on them, the more likely we are to pick the right metric for measuring their compactness. Conversely, selecting the right metric for measuring compactness will improve our understanding of geographic shapes and our analytical ability to explain the forces acting on them. This essay, building on the new possibilities for measuring geographic shapes using GIS software, is but a step in making the compactness of geographic shapes better understood. We hope that it paves the way not simply for the development of new metrics for measuring geographic shapes, but for a better understanding of how to interpret these metrics correctly and put them to good use.

Acknowledgements

The authors would like to thank Michael Batty, Alain Bertaud, Robert Buckley, Ralph Gakenheimer, Michael Heller, James Hurd Jr., Geoffrey Hyman, Stephen Malpezzi, David Potere, Annemarie Schneider, Waldo Tobler and Shue Tuck Wong for their useful comments and suggestions as well as for their encouragement and support.

References

ANGEL, S., SHEPPARD, S. C., CIVCO, D. L., BUCKLEY, R., CHABAEVA, A., GITLIN, L. KRALEY, A., PARENT, J., and PERLIN, M. 2005 *The Dynamics of Global Urban Expansion*, Transport and Urban Development Department, The World Bank, (Washington, DC: The World Bank), September (Available at: http://go.worldbank.org/58A0YZVOV0, Accessed 16 October 2009)

BLAIR, D. J., and BISS, T. H. 1967 'The Measurement of Shape in Geography: An Appraisal of Methods and Techniques' *Bulletin of Quantitative Data for Geographers* 11

BOYCE, R. R., and CLARK, W. A. V. 1964 'The concept of shape in Geography' *The Geographical Review* 54, 561–572

BURCHFIELD, M., OVERMAN, H. G., PUGA, D., and TURNER, M. E. 2006 'Causes of sprawl: a portrait from space' *Quarterly Journal of Economics* 121(2), 587–633

DE SMITH, M. J. 2004 *Distance and Path: The Development, interpretation and Application of Distance Measurement in Mapping and Spatial Modelling.* Unpublished PhD dissertation, Department of Geography, University College, University of London (Available at: http://www.desmith.com/MJdS, Accessed 19 October 2009)

DE SMITH, M. J., GOODCHILD, M. F., and LONGLEY, P. A. 2007 *Geospatial analysis: A Comprehensive Guide to Principles, Techniques and Software Tools* (Leicester: Matador)

EHRENBERG, K. 1892 'Studies on the measurement of horizontal shapes of areas' *Vehandl. der. phys.-mediz Gesellschaft zu Würzburg*, Neue Folge 25, n.p., quoted in Y. S. Frolov 'Measuring of Shape of Geographical Phenomena: a History of the Issue' *Soviet Geography: Review and Translation* 16, 1975, 676–687

FROLOV, Y. S. 1975 'Measuring of shape of geographical phenomena: a history of the issue', *Soviet Geography: Review and Translation* 16, 676–687

GUSTAFSON, E. J. 1998 'Quantifying landscape spatial pattern: what is the state of the art?' *Ecosystems* 1, 143–156

HILBERT, D., and COHN-VÓSSEN, S. 1952 'Eleven properties of the sphere', in *Geometry and the Imagination*, trans. P. Nemenyi (New York: Chelsea), 215–232

HORN, D. L., HAMPTON, C. R., and VANDENBERG, A. J. 1993 'Practical application of district compactness' *Political Geography* 12, 103–120

KENDALL, M. G., and MORAN, P. A. P. 1963 *Geometrical Probability* (New York: Hafner Publishing Company)

MACEACHREN, A. M. 1985 'Compactness of geographic shape: comparison and evaluation of measures' *Geografisca Annaler*, Series B, 67(1), 53–67

McGARIGAL, K., and MARKS, B. J. 1995 *Fragstats: Spatial Pattern Analysis Program for Quantifying Landscape Structure* (Available at: http://www.fs.fed.us/pnw/pubs/pnw_gtr351.pdf, Accessed 16 October 2009)

MILNE, B. T. 1988 'Measuring the fractal geometry of landscapes' *Applied Mathematics and Computation* 27, 67–79

NAGEL, DR. 1835 'On Coastal Development of Continents' *Annal von Berghaus 12*, n.p., quoted in Y. S. Frolov 'Measuring of Shape of Geographical Phenomena: a History of the Issue' *Soviet Geography: Review and Translation* 16, 1975, 676–687

POLSBY, D. D., and POPPER, R. D. 1991 'The third criterion: Compactness as a procedural safeguard against gerrymandering' *Yale Law and Policy Review* 9, 301–353

RITTER, C. 1822 *Die Erdkunde im Verhältniss zur Natur und Geschichte des Menschen, oder allgemeine vergleichende Geographie*, 2nd edition, Part I, Book 1, Berlin, quoted in Y. S. Frolov 'Measuring of Shape of Geographical Phenomena: a History of the Issue' *Soviet Geography: Review and Translation* 16, 1975, 676–687

ROHRBACH, C. 1890 'On Mean Frontier Distances', *Petermanns Geogr. Mitteil* 36, n.p., quoted in Y. S. Frolov 'Measuring of Shape of Geographical Phenomena: a History of the Issue' *Soviet Geography: Review and Translation* 16, 1975, 676–687

SULLIVAN, L. 1896 'The tall office building artistically considered' *Lippincott's Monthly Magazine* March, 403–409.

THOMPSON, D. W. 1952 *On Growth and Form*, 2nd edition, vol. 1, (Cambridge: Cambridge University Press)

THÜNEN, J. H. VON 1966 *Isolated state; an English edition of Der isolierte Staat*, ed. P. G. Hall, trans. Carla Wartenburg (Oxford: Pergamon Press), quoted in Y. S. Frolov 'Measuring of Shape of Geographical Phenomena: a History of the Issue' *Soviet Geography: Review and Translation* 16, 1975, 676–687

TIKHOMIROV, V. M. 1990 *Stories about Maxima and Minima*, transl. by Abe Shenitzer (Providence, RI: American Mathematical Society)

US SUPREME COURT 1996 *Bush v. Vera*, 517 US 952 (O'Connor, J., opinion of the Court)

—. 2004 *Vieth v. Jubelirer*, 541 US 267, (Souter, J., dissenting)

—. 2006a *League v. Perry*, 548 US 399 (Kennedy, J., opinion of the Court)

—. 2006b *League v. Perry*, 548 US 399 (Roberts, J., dissenting)

WILDGREN, J. K. 1995 'Fractal Geometry and the Boundaries of Voting Districts' in *Spatial and Contextual Models in Political Research*, ed. M. Eagles (London: Taylor & Francis), 107–126

WHITTINGTON, G., BEAVON, K. S. O., and MABIN, A. S. 1972 *Compactness of Shape: Review, Theory, and Application* (Johannesburg:

Department of Geography and Environmental Studies, University of Witwatersand)

WURM, S. A., and HATTORI, S., eds. 1981 *Language Atlas of the Pacific Area* Pacific Linguistics Series C No. 66 (Canberra: Department of Linguistics, Research School of Pacific Studies, Australian National University)

YOUNG, H. P. 1988 'Measuring the compactness of legislative districts' *Legislative Studies Quarterly* 13(1), 105–115

Mathematical Appendix: Calculation of the Compactness Metrics

To calculate the various indices, a Python program was developed that can be run through ESRI's ArcGIS software (Environmental Systems Research Institute, www.esri.com). The 'Shape Metrics Tool' is fully documented and available for general use. It requires input data to be in an ESRI feature class file and ArcGIS v9.2 or later to operate (www.clear.uconn.edu/tools).

In the metrics below, the Proximate Center (x_c, y_c) is the center of gravity of a shape with area A, and perimeter P_A. The radius of the shape's equal area circle is calculated as $R_A = \sqrt{A/\pi}$.

The cohesion index

Calculate the average inter-point distance-squared d_{IS} between a sample of m random points in the shape

$$d_{IS} = (1/n^2) \cdot \sum_{i=1}^{m} \sum_{j=1}^{m} (x_i - x_j)^2 + (y_i - y_j)^2. \tag{1.1}$$

The authors have found the average inter-point distance-squared in a circle of area A and radius R_A to be

$$d_{IA} = R_A^2. \tag{1.2}$$

Because $R_A^2 = A/\pi$, we have $d_{IA} = A/\pi$. The formula for calculating the Cohesion Index I_C is therefore

$$I_C = d_{IA}/d_{IS} = A/\pi\, ds. \tag{1.3}$$

The proximity index

Calculate the average distance to the Proximate Centre of the shape, d_{CS}:

$$d_{CS} = (1/n) \sum_{i=1}^{n} (x_i - x_c)^2 + (y_i - y_c)^2. \tag{2.1}$$

The average distance d_{CA} to the centre in a circle of radius R_A is given in the literature (de Smith 2004, 207) as

$$d_{CA} = 2R_A/3. \tag{2.2}$$

Because $R_A = \sqrt{(A/\pi)}$, we have $d_{CA} = (2\sqrt{(A/\pi)})/3$. The formula for calculating the Proximity Index I_Y is therefore

$$I_Y = d_{CA}/d_{CS} = (2\sqrt{(A/\pi)})/3d_{CS}. \tag{2.3}$$

The exchange index

Draw the equal-area circle about the Proximate Centre C_P and calculate the area of overlap O_S, of the equal-area circle and the shape. The formula for calculating the Exchange Index I_X is

$$I_X = O_S/A. \tag{3.1}$$

The perimeter index

Find the perimeter P of the shape. The formula for calculating the Exchange Index I_P is

$$I_P = P_A/P = (2\sqrt{\pi A})/R. \tag{4.1}$$

The fullness index

Calculate the radius r_A of a small neighbourhood in the shape, so that $\pi r_A^2 = A/100$.

$$r_A = \sqrt{(A/100\pi)}. \tag{5.1}$$

Find the average fullness of the shape, F_S, as the average of the fullness F_i of a small circle of radius r_A about the centre of every pixel i in the shape

$$F_S = \left(\sum_{i=1}^{m} F_i\right) \Big/ n. \tag{5.2}$$

Using ArcGIS, we established that for any circle, the average fullness equals 0.957. The formula for calculating the Fullness Index I_F is therefore

$$I_F = F_S/0.957. \tag{5.3}$$

The depth index

Calculate the average straight-line distance d_{PS} from the centres of all pixels to the outer periphery of the shape.

From equation 2.2 we can infer that the average distance to the periphery of the equal-area circle is

$$d_{PA} = R_A/3. \tag{6.1}$$

Because $R_A = \sqrt{}(A/\pi)$, we have $d_{PA} = \sqrt{}(A/\pi)/3$. The formula for calculating the Depth Index I_H is therefore

$$I_H = d_{PS}/d_{PA} = 3ds/\sqrt{(A/\pi)}. \tag{6.2}$$

The dispersion index

Given a shape with a given centre C, we can find the Average Distance Circle of radius R_{AD} and an area A_{AD} by successive approximations, looking for a circle where the area of the shape outside it, A_O, is equal to the area not in the shape inside it, A_I. If C is not given, we use the Proximate Centre of the shape C_P as the centre C of the circle. We know that

$$A_O = A_I. \tag{7.1}$$

The formula for calculating the Dispersion Index I_D is therefore

$$I_D = (A - A_O)/A. \tag{7.2}$$

The reader can ascertain that the circle attains a score of 1 on the Dispersion Index, and that $0 \leq I_D \leq 1$.

To ascertain that the chosen circle is, in fact, an Average-Distance Circle, observe that the average radial distance d of the perimeter of the shape from the perimeter of the Average-Distance Circle is

$$d = (A_O + A_I)/2\pi R = A_O/\pi R = A_I/\pi R. \tag{7.3}$$

Clearly, the average of $R + d$ and $R - d$ is R. We therefore also have

$$I_D = (A - A_O)/A = (\pi R^2 - d\pi R)/\pi R^2 = 1 - d/R. \tag{7.4}$$

The range index

Find the diameter of the smallest circle circumscribing the given shape, D_S, using ArcGIS. The diameter of the smallest circle encompassing the equal-area circle is its own diameter, $D_A = 2R_A$. Because $R_A = \sqrt{}(A/\pi)$, we have $D_A = 2\sqrt{}(A/\pi)$. The formula for calculating the Range Index I_R is therefore:

$$I_R = D_A/D_S = 2\sqrt{(A/\pi)}/D_S. \tag{8.1}$$

The girth index

Find the innermost point in the shape by successive approximations. The radius of the largest circle circumscribed by the shape, R_S, will be equal to the distance of the innermost point from its perimeter. The radius of the largest circle fully encompassed by the equal-area circle is its own radius, R_A.

$$I_G = R_S/R_A. \tag{9.1}$$

Because $R_A = \sqrt{}(A/\pi)$, the formula for calculating the Girth Index I_G is

$$I_G = R_S/\sqrt{(A/\pi)}. \tag{9.2}$$

The traversal index

The Traversal Index can only be calculated for shapes that are formed of a single contiguous patch. The average distance along interior paths between points on the perimeter of the shape, d_{TS}, needs to be calculated using one of a number of possible algorithms for calculating distances on interior paths.

It can be shown using simple integration that for a circle of area A and radius R, the average distance d_{TA} between points along its perimeter is

$$d_{TA} = 4R/\pi = 1.2732R. \tag{10.1}$$

Because $R = A/\pi$, we have $d_{TA} = 1.2732\sqrt{}A/\pi)$. The formula for calculating the Traversal Index I_T is therefore

$$I_T = d_{TA}/d_{TS} = 1.2732\sqrt{A/\pi}\, ds. \quad (10.2)$$

The detour index

Find the Convex Hull corresponding to the shape using any of a number of algorithms developed to calculate it in ArcGIS, and calculate its perimeter, P_{CH}. The detour index I_U is the ratio of the perimeter of the equal-area circle and the perimeter of the convex hull:

$$I_U = (2\sqrt{\pi A})/P_{CH}. \quad (11.1)$$